Circuit Quantum Electrodynamics

Superconducting platform

(fourth Lecture)

Covering: basic concepts, measurement techniques, implementations, qubit approaches, current trends

With figures and slides borrowed from

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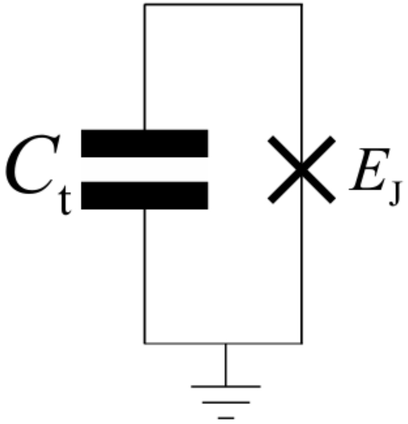
Transmon Limit

Review: The Cooper-pair box (CPB) and transmon



- The *CPB* consists of a parallel combination of a capacitor and a Josephson junction.

- The circuit looks a lot like the *LC* oscillator!

- However, there is a key difference: the inductor of the *LC* oscillator is here replaced by a Josephson junction.

- The circuit for a *transmon* is identical to that of the CPB!

- CPBs and transmons differ *only* in the characteristic values of the capacitor and junction.

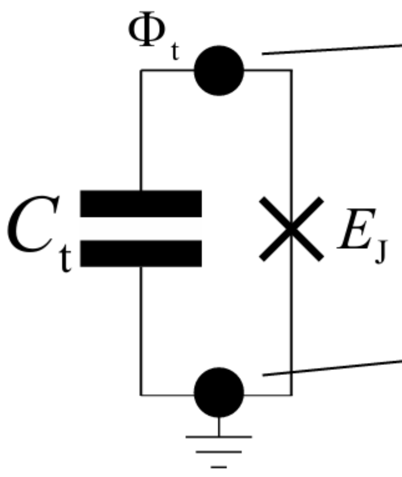
- These circuits **are not truly quantum two-level systems**.
Rather, they are quantum *many*-level systems.
For this reason, they are often referred to as *artificial atoms*.

Review: The Cooper-pair box (CPB) and transmon

Construct the classical Lagrangian:

$$\mathfrak{L} = E_\mathrm{e} - E_\mathrm{m}$$

$$E_\mathrm{e} = \frac{1}{2} C_\mathrm{t} \dot{\Phi}_\mathrm{t}^2 \qquad E_\mathrm{m} = -E_\mathrm{J} \cos\left(2\pi \frac{\Phi_\mathrm{t}}{\Phi_o}\right)$$



Find the variable conjugate to flux:

$$Q_\mathrm{t} = \frac{d\mathfrak{L}}{d\dot{\Phi}_\mathrm{t}} = C_t \dot{\Phi}_\mathrm{t}$$

Construct classical Hamiltonian:

$$H(\Phi_\mathrm{t}, Q_\mathrm{t}) = \dot{\Phi}_\mathrm{t} Q_\mathrm{t} - \mathfrak{L} = -E_\mathrm{J} \cos\left(2\pi \frac{\Phi_\mathrm{t}}{\Phi_0}\right) + \frac{Q_\mathrm{t}^2}{2C_\mathrm{t}}$$

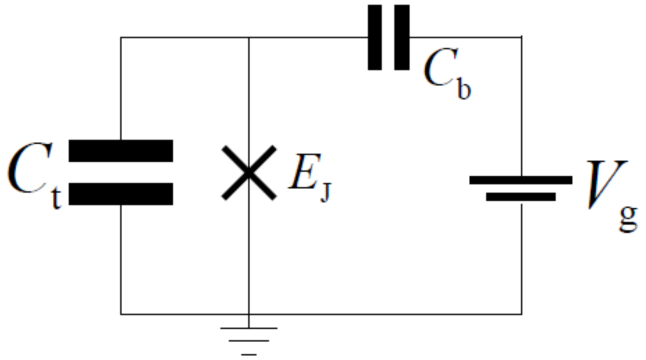
Construct quantum Hamiltonian:
replace variables by operators satisfying canonical commutation relations:

$$\longrightarrow \hat{H} = -E_\mathrm{J} \cos\left(2\pi \frac{\hat{\Phi}_\mathrm{t}}{\Phi_o}\right) + \frac{\hat{Q}_\mathrm{t}^2}{2C_\mathrm{t}} \qquad \left[\hat{\Phi}_\mathrm{t}, \hat{Q}_\mathrm{t}\right] = i\hbar$$

M. Devoret, Les Houches Session LXIII (1995)
Koch *et al.*, Phys. Rev. A 76, 042319 (2007)

Hamiltonian of CPB with voltage bias

$$\hat{H} = -E_J \cos\left(2\pi \frac{\hat{\Phi}_t}{\Phi_o}\right) + \frac{\left(\hat{Q}_t + C_b V_g\right)^2}{2\left(C_t + C_b\right)}$$



Is commonly written using other variables:

$$\hat{\varphi}_t \equiv 2\pi \frac{\hat{\Phi}_t}{\Phi_o} \qquad \hat{N}_t \equiv -\frac{\hat{Q}_t}{2e} \qquad N_g \equiv \frac{C_b V_g}{2e} \qquad E_C \equiv \frac{e^2}{2\left(C_t + C_b\right)}$$

Phase operator, Cooper-pair number operator: Quantum operators

Number offset, Charging energy: These are numbers (not operators!)

$$\hat{H} = -E_J \cos\left(\hat{\varphi}_t\right) + 4E_C\left(\hat{N}_t - N_g\right)^2 \qquad \left[\hat{\varphi}_t, \hat{N}_t\right] = -i$$

Hamiltonian in charge basis

$$\hat{H} = \sum_{N=-\infty}^{\infty} -\frac{E_J}{2}\left(|N\rangle\langle N+1| + |N+1\rangle\langle N|\right) + 4E_C\left(N - N_g\right)^2 |N\rangle\langle N|$$

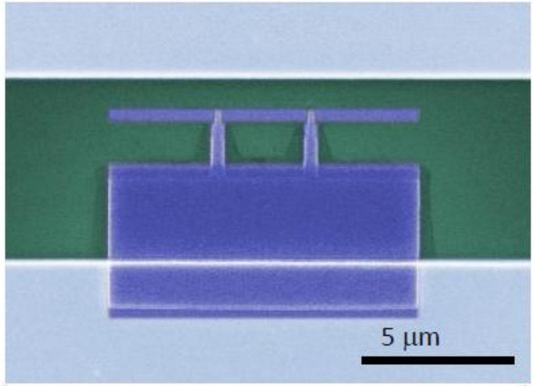
CPB and transmon regimes

$$\hat{H} = \sum_{N=-\infty}^{\infty} -\frac{E_J}{2}\left(|N\rangle\langle N+1| + |N+1\rangle\langle N|\right) + 4E_C\left(N - N_g\right)^2 |N\rangle\langle N|$$

CPB and Transmon differ only in the regimes of Josephson and charging energies used.

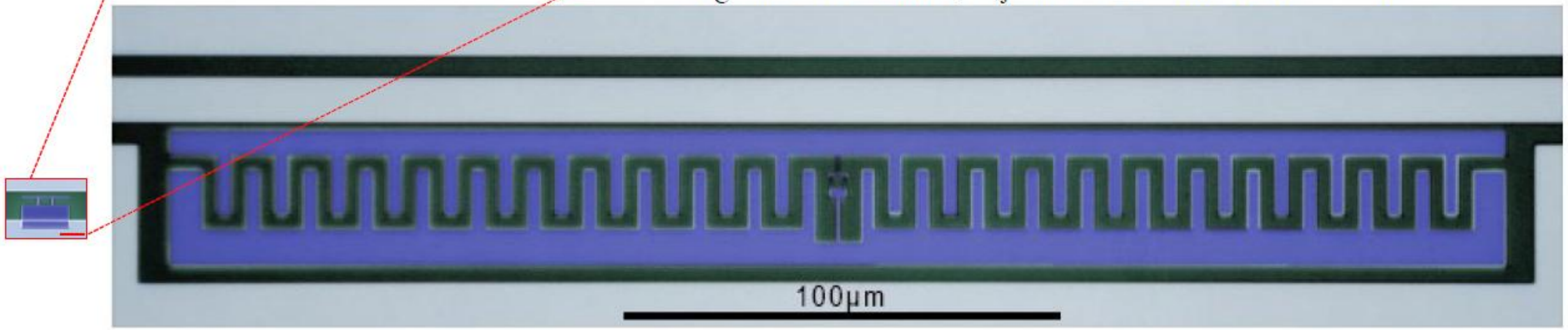
Cooper-Pair Box $\frac{E_J}{E_C} \sim 1$

$E_C / h \sim E_J / h \sim 5$ GHz



Transmon $\frac{E_J}{E_C} >\sim 30$

$E_C / h \approx 0.3$ GHz; $E_J / h \sim 10-30$ GHz



$E_J = 0$

$$\hat{H} = \sum_{N=-\infty}^{\infty} -\frac{E_J}{2}\left(|N\rangle\langle N+1| + |N+1\rangle\langle N|\right) + 4E_C\left(N-N_g\right)^2 |N\rangle\langle N|$$

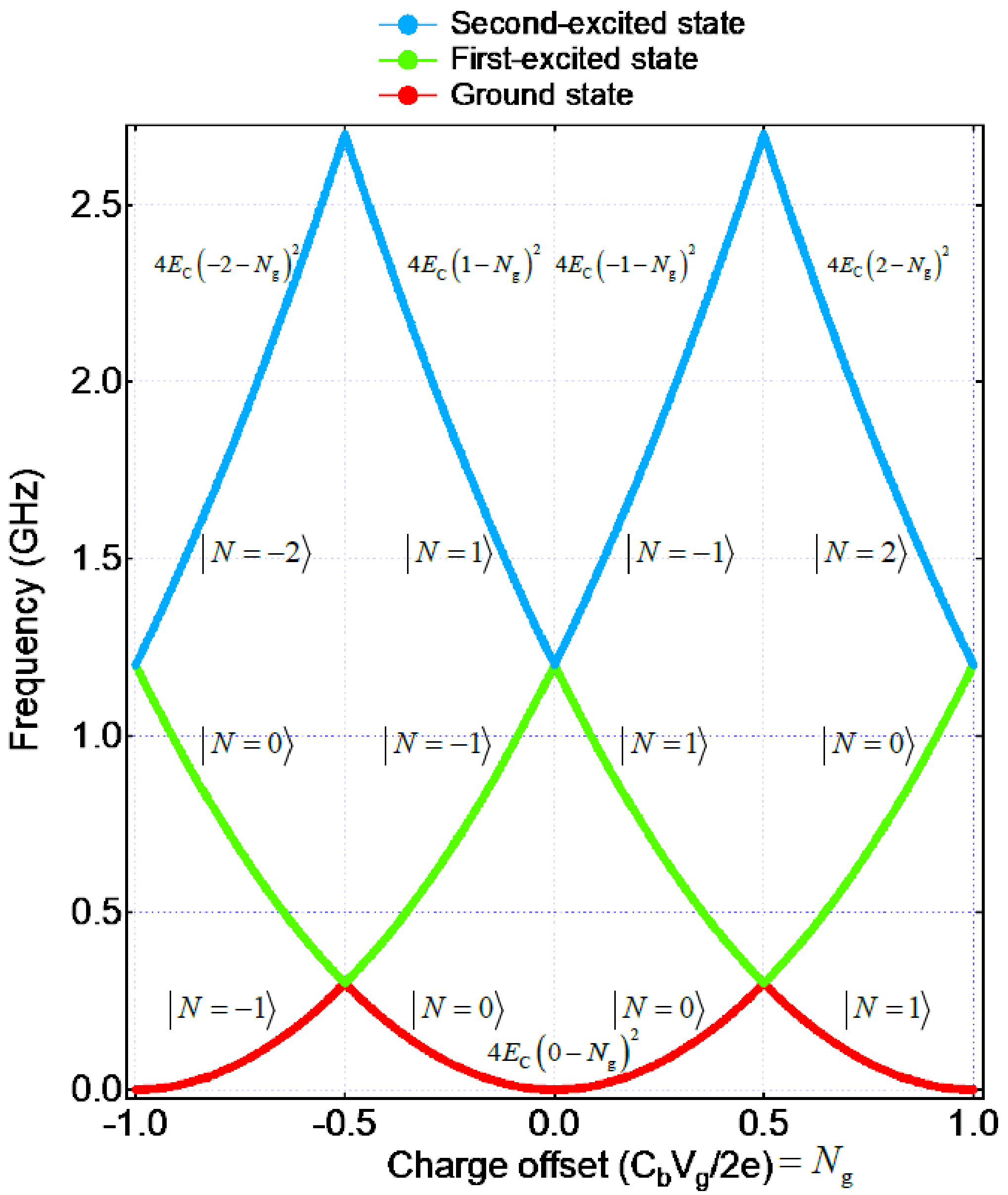
0

- The Hamiltonian is diagonal in the charge basis.
- What are the eigenstates? $|N\rangle$
- What are the eigenvalues? $4E_C\left(N - N_g\right)^2$

E_J = 0 (Energy spectrum)

With the Josephson coupling completely turned off, the eigenstates are simply the charge states $|N\rangle$.

Their ordering in energy depends on the value of the charge offset.



$E_C / h = 0.3$ GHz

$E_J = 0$

Second-excited state

First-excited state

Ground state

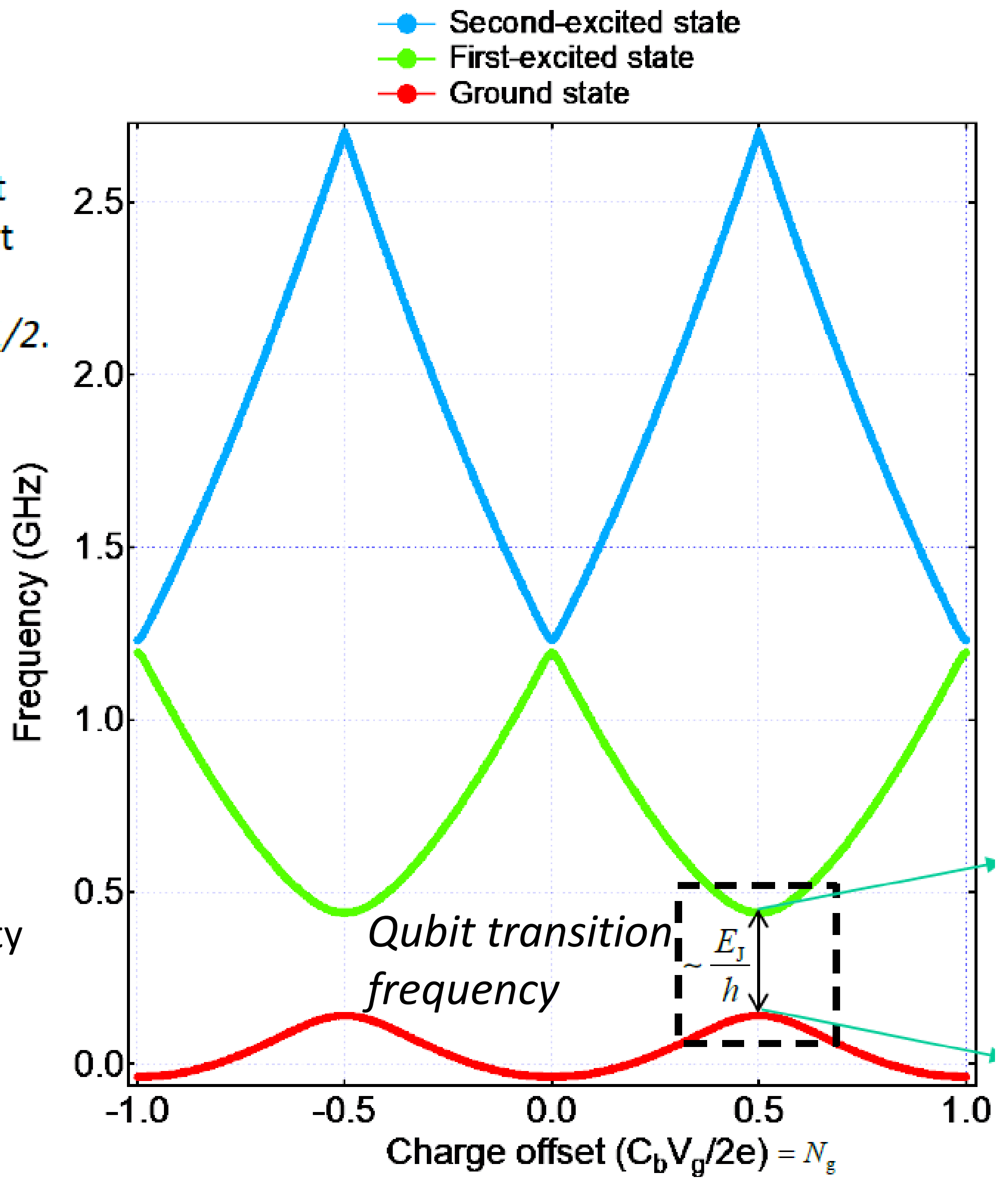
courtesy Di Carlo (TU Delft)

CPB: $E_J = E_C$

The CPB quantum bit consists of the lowest two energy states at charge offset N_g=*1/2*.

Large anharmonicity

$\alpha \equiv f_{12} - f_{01}$



$E_C / h = 0.3$ GHz

$E_J / h = 0.3$ GHz

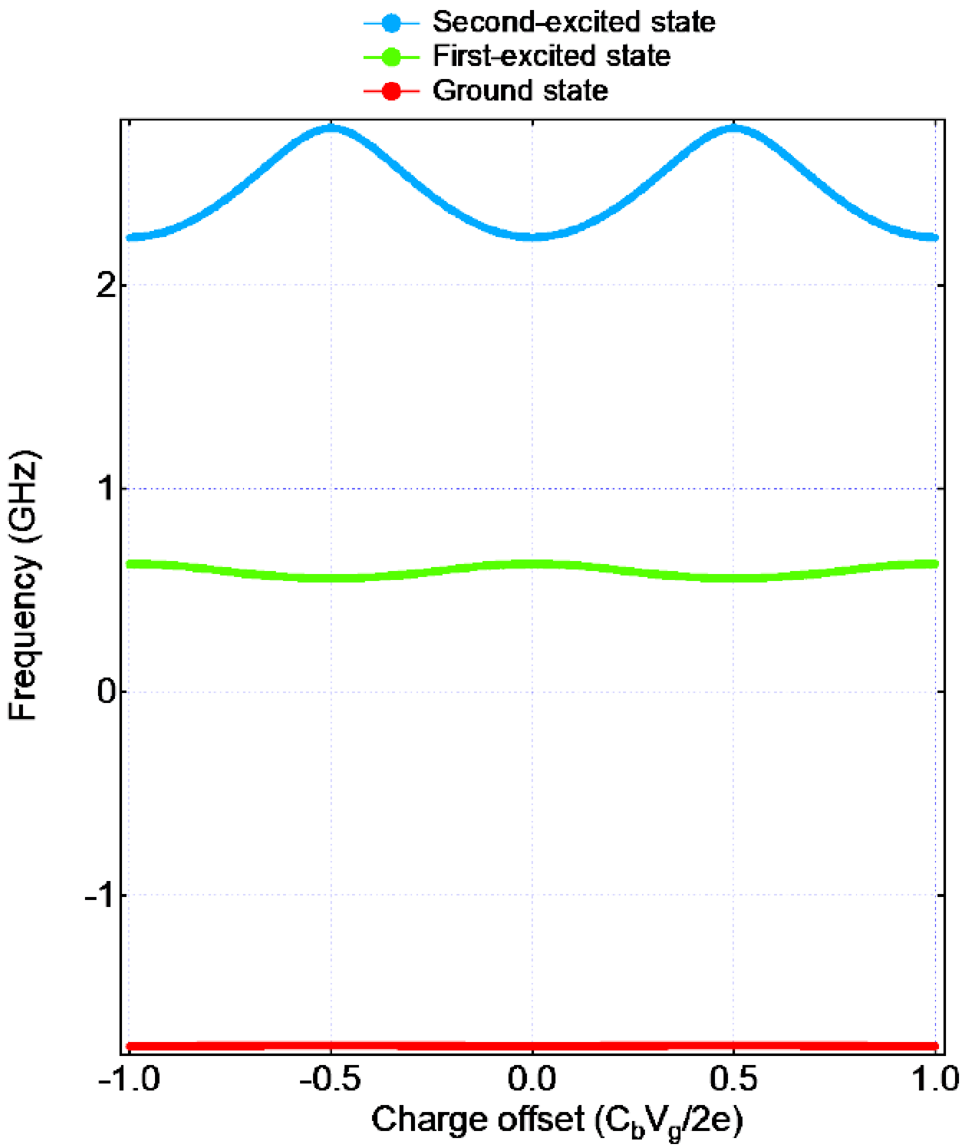
At this charge offset, the ground and first excited states are, to a good approximation, the symmetric and antisymmetric maximal superposition of charge states $|N=0\rangle$ and $|N=1\rangle$.

$$|\psi_1\rangle \approx \frac{1}{\sqrt{2}}|N=0\rangle - \frac{1}{\sqrt{2}}|N=1\rangle$$

$$|\psi_0\rangle \approx \frac{1}{\sqrt{2}}|N=0\rangle + \frac{1}{\sqrt{2}}|N=1\rangle$$

courtesy Di Carlo (TU Delft)

Intermediate regime: $E_J > E_C$



$E_C / h = 0.3$ GHz

$E_J / h = 3$ GHz

courtesy Di Carlo (TU Delft)

Transmon regime: $E_J >> E_C$

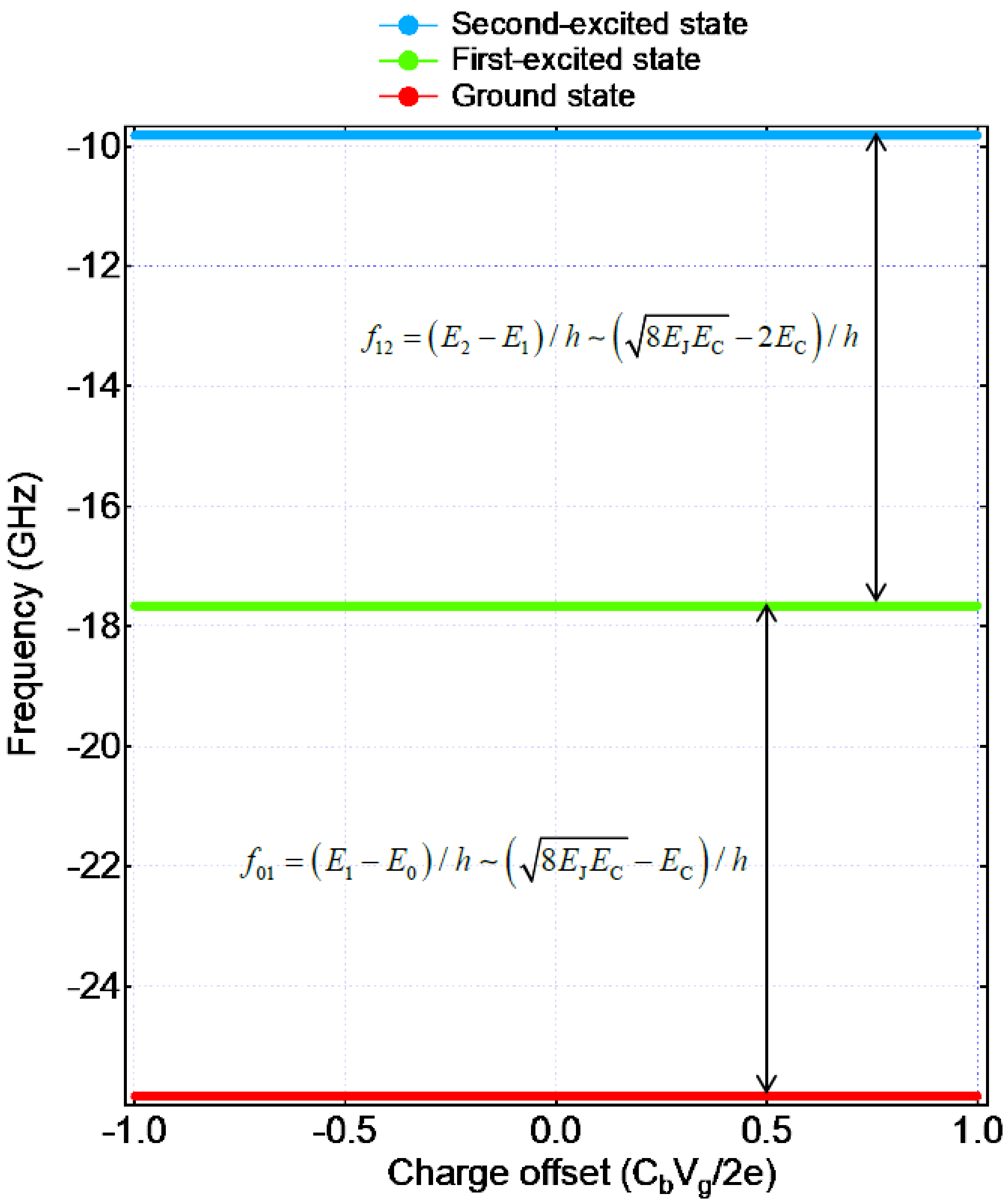
$E_C / h = 0.3$ GHz

$E_J / h = 30$ GHz

In the transmon regime, the energy levels become insensitive to the charge offset.

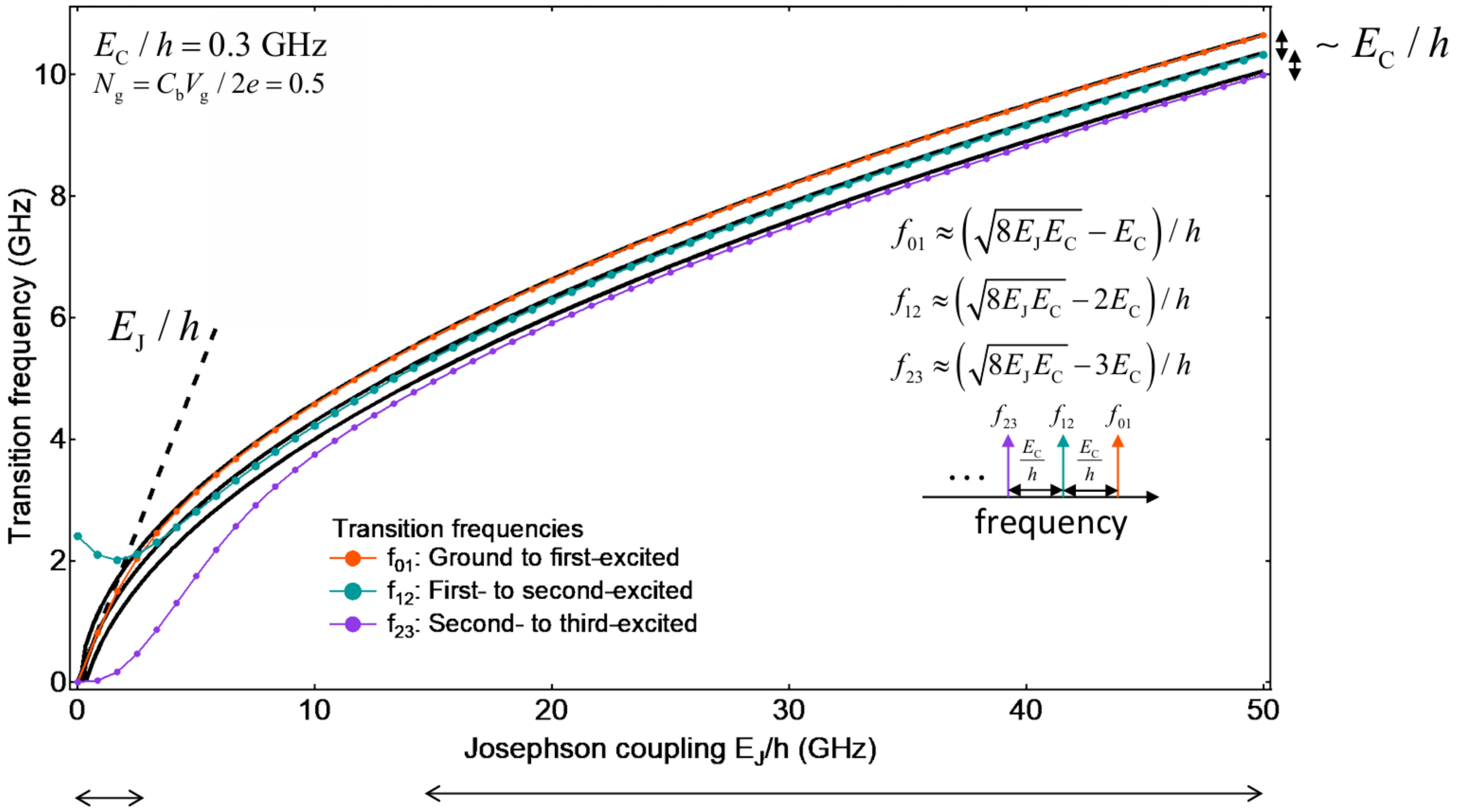
The anharmonicity is reduced but not fully eliminated

$\alpha \equiv f_{12} - f_{01} \sim -E_C / h$



courtesy Di Carlo (TU Delft)

Transition frequencies as a function of EJ

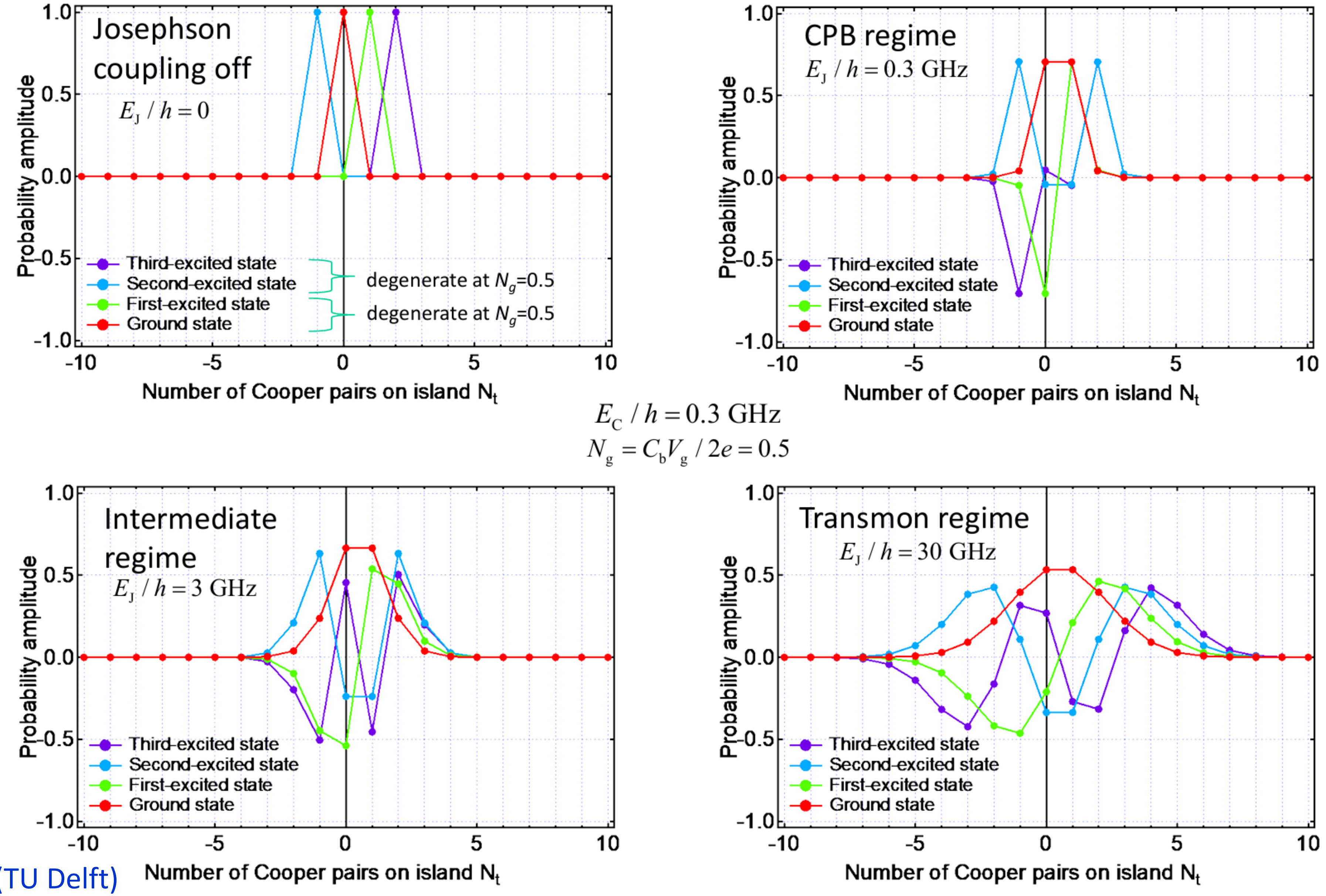


CPB regime
Very anharmonic but
charge sensitive spectrum

Transmon regime
charge insensitive and
weakly anharmonic spectrum

courtesy Di Carlo (TU Delft)

Wavefunction in the charge basis



$$E_C / h = 0.3 \text{ GHz}$$
$$N_g = C_b V_g / 2e = 0.5$$

courtesy Di Carlo (TU Delft)

Important messages

● The charge qubit circuit consists of a parallel combination of a capacitor and a Josephson junction. Using a Josephson junction as the inductive element (as opposed to a inductor in the *LC* oscillator) makes the energy spectrum *anharmonic*.

● This quantum system has multiple levels, *not just two*. The anharmonic spectrum helps to confine the dynamics two a two-level subspace (we will see this later). The two-level subspace used to define an *effective* qubit consists of the two lowest-energy levels.

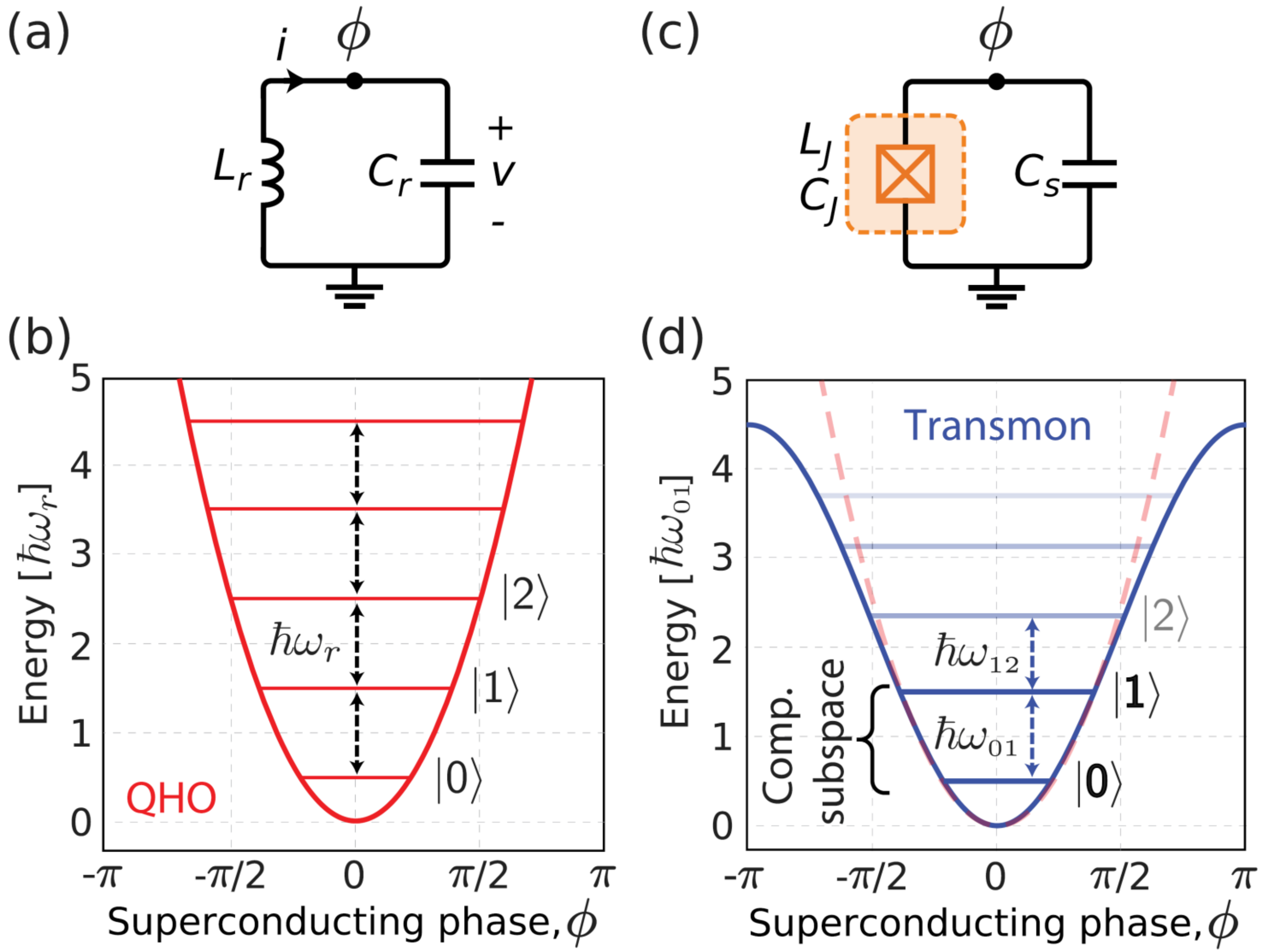
● The two relevant energy scales of the charge-qubit Hamiltonian are the Josephson coupling energy E_J and the charging energy E_C . In the Cooper-pair box regime, $E_J / E_C \sim 1$.
In the transmon regime, $E_J / E_C >\sim 30$.

● In the CPB regime, a voltage bias is required. The qubit transition frequency at the typical bias point is approximately E_J / h.

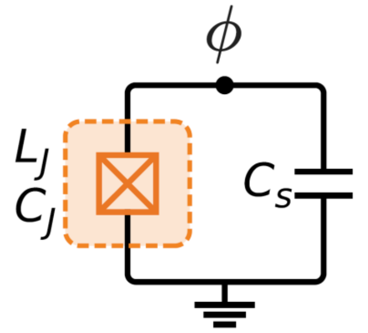
● In the transmon regime, the energy levels are insensitive to voltage bias (and also to charge noise!). The qubit transition frequency is approximately $\left(\sqrt{8E_J(\Phi_{ext})E_C} - E_C\right)/h$.
The *anharmonicity* is reduced but not eliminated, it is approximately $-E_C / h$.

● Using two Josephson junctions in parallel (as opposed to just one) allows tuning the Josephson coupling energy and correspondingly the transition frequencies with an applied magnetic flux.

Transmon as an anharmonic oscillator



Transmon as an anharmonic oscillator



- Step 1: Write down Lagrangian

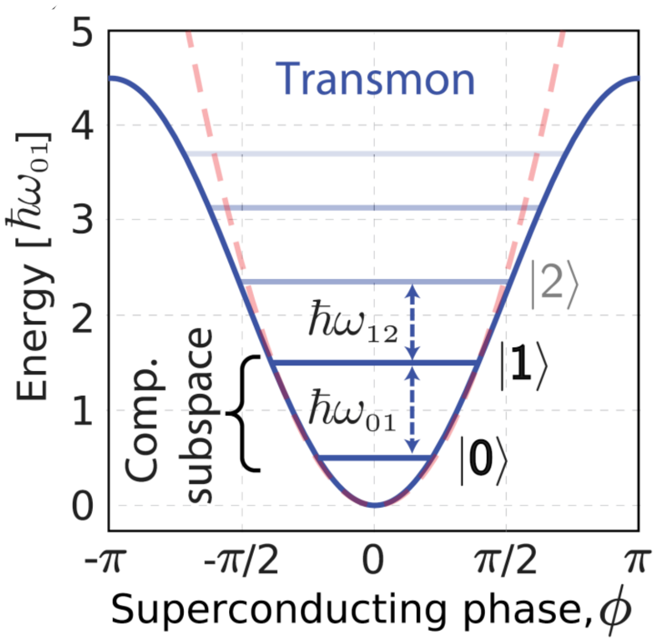
$$\mathcal{L} = E_{cap} - E_{ind} = \frac{1}{2} C\,\dot{\Phi}^2 + E_J \cos\left(2\pi \frac{\Phi}{\Phi_0}\right)$$
$$C = C_S + C_J$$

- Step 2: Find conjugate variable

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi}$$

- Step 3: Calculate classical Hamiltonian

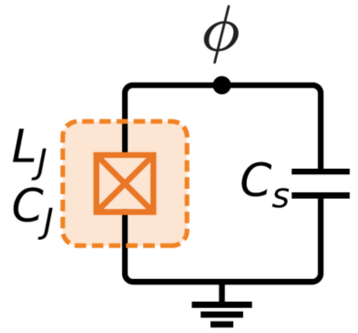
$$\mathcal{H}(\Phi, Q) = C\dot{\Phi}^2 - \mathcal{L}(\Phi, \dot{\Phi}) = \frac{1}{2C} Q^2 - E_J \cos\left(2\pi \frac{\Phi}{\Phi_0}\right)$$



- Step 4: Quantize the Hamiltonian

$$[\hat{\Phi}, \hat{Q}] = i\hbar \qquad H = -E_J \cos\left(2\pi \frac{\hat{\Phi}}{\Phi_0}\right) + \frac{1}{2C}\hat{Q}^2 = -E_J \cos\left(2\pi \frac{\hat{\Phi}}{\Phi_0}\right) + 4E_C\, n^2 \qquad n = Q/2e$$

Transmon as an anharmonic oscillator

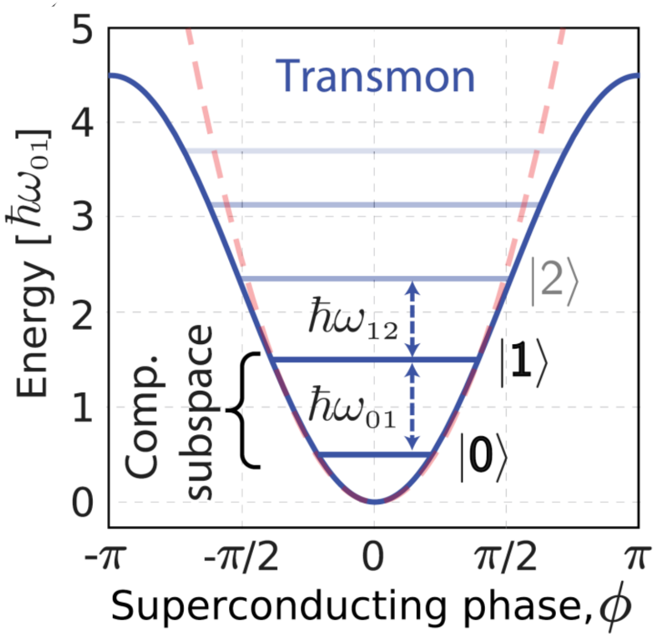


- What if Φ is very very small?
  - Question: when does that happen?

$$H = -E_J \cos\left(2\pi \frac{\hat{\Phi}}{\Phi_0}\right) + \frac{1}{2C}\hat{Q}^2 \approx -E_J\left(1 - 0.5\left(2\pi\frac{\hat{\Phi}}{\Phi_0}\right)^2\right) + \frac{1}{2C}\hat{Q}^2$$

$$H = \frac{E_J}{2}\left(\frac{2\pi}{\Phi_0}\right)^2 \hat{\Phi}^2 + \frac{1}{2C}\hat{Q}^2 = \frac{E_J}{2}\hat{\varphi}^2 \quad + 4E_C\,\hat{n}^2$$

Harmonic Oscillator

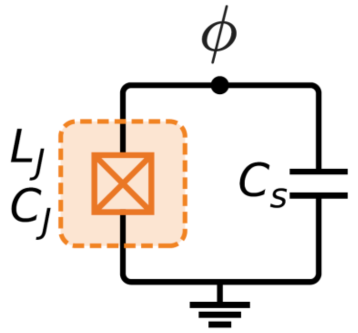


- What if Φ is small?

$$H = \frac{E_J}{2}\left(\frac{2\pi}{\Phi_0}\right)^2 \hat{\Phi}^2 - \frac{E_J}{24}\left(\frac{2\pi}{\Phi_0}\right)^4 \hat{\Phi}^4 + \frac{1}{2C}\hat{Q}^2$$

Non-linear energy term

Transmon as an anharmonic oscillator



- Consider the Hamiltonian term:

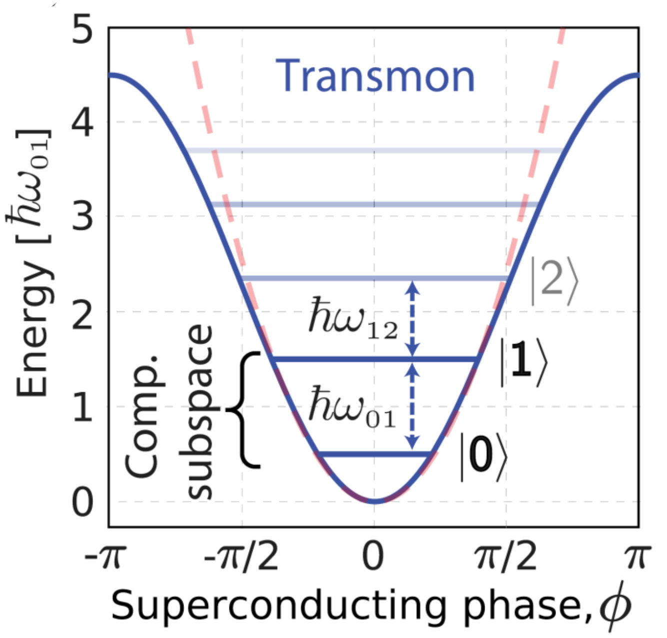
$$-\frac{E_J}{24}\left(\frac{2\pi}{\Phi_0}\right)^4 \hat{\Phi}^4$$

- Rewrite this term to:

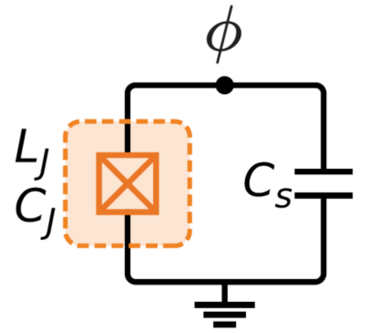
$$-Ka^\dagger a^\dagger aa - \delta a^\dagger a + \Delta_0$$

- Hints:
  - Keep only "energy preserving" terms
    (same number of creation and annihilation operators)

$$\hat{\Phi} = \sqrt{\frac{\hbar}{2C\omega}}(a^\dagger + a), \qquad [a, a^\dagger] = 1$$



Transmon as an anharmonic oscillator



- Hamiltonian

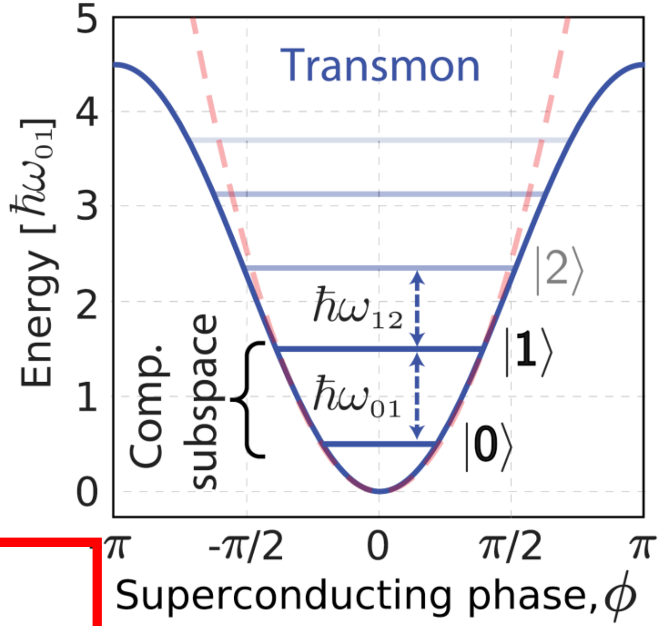
$$H = \frac{E_J}{2}\left(\frac{2\pi}{\Phi_0}\right)^2 \hat{\Phi}^2 + \frac{1}{2C}\hat{Q}^2 - \frac{E_J}{24}\left(\frac{2\pi}{\Phi_0}\right)^4 \hat{\Phi}^4$$

$$H_{HO} = \frac{E_J}{2}\left(\frac{2\pi}{\Phi_0}\right)^2 \hat{\Phi}^2 + \frac{1}{2C}\hat{Q}^2 = \frac{E_J}{2}\hat{\varphi}^2 \quad + 4E_C\,\hat{n}^2$$

- Conjugate variable $[\hat{\varphi}, \hat{n}] = i$

By rewriting $\hat{\varphi} = \left(\frac{2E_c}{E_J}\right)^{\frac{1}{4}} (a^\dagger + a)$, $\quad \hat{n} = \frac{i}{2}\left(\frac{E_J}{2E_C}\right)^{\frac{1}{4}} (a^\dagger - a)$

where $a^\dagger$ and a are the rising and lowering operators diagonalizing H_{HO}

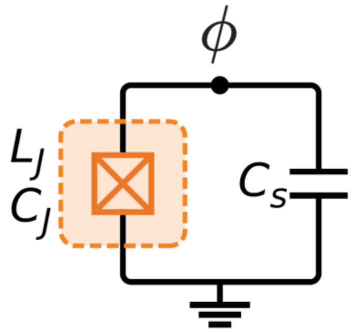


- Approximate Hamiltonian

$$H \approx \sqrt{8E_C E_J}\, a^\dagger a - \frac{E_c}{12}(a^\dagger + a)^4 \approx \hbar\omega_q a^\dagger a - \frac{E_c}{2} a^\dagger a^\dagger a a$$

with $\hbar\omega_q = \sqrt{8E_C E_J} - E_C$

Anharmonic energy spectrum



- With the Hamiltonian

$$\hbar\omega = \sqrt{8E_J E_C}$$

$$H = \hbar\omega a^\dagger a - E_c a^\dagger a - \frac{E_c}{2} a^\dagger a^\dagger a a =$$

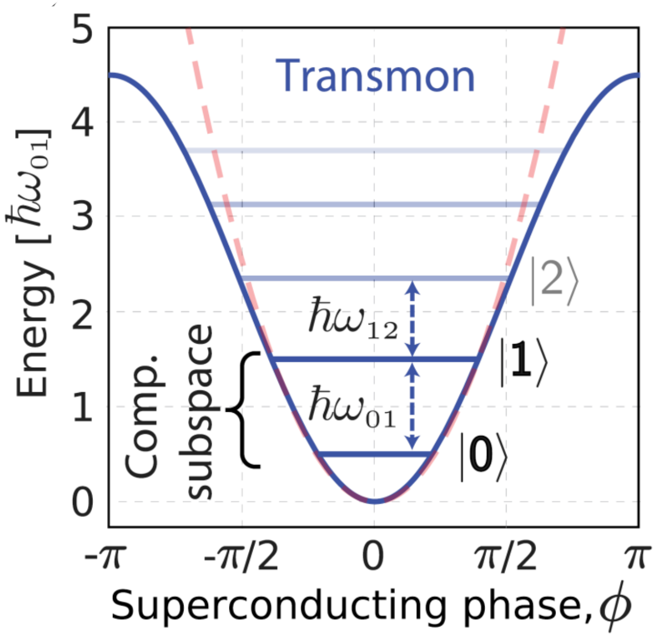
$$= \hbar\omega_T a^\dagger a - \frac{\hbar\alpha}{2} a^\dagger a^\dagger a a$$

$$\boldsymbol{\hbar\omega_T = \sqrt{8E_J E_C} - E_C} \qquad \boldsymbol{\hbar\alpha = E_C}$$

Transmon frequency — Transmon anharmonicity

- Energies:

$$E_1 - E_0 = \hbar\omega - E_c$$
$$E_2 - E_1 = \hbar\omega - 2E_c$$



- Because $E_2 - E_1 \neq E_1 - E_0$ we can use as a qubit